A THEOREM OF DYNAMICS

(ODNA TEOREMA DINAMIKI)

PMM Vol.28, № 6, 1964, pp.1138-1139

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(Received July 10, 1964)

We are considering a system of two rigid bodies S_1 and S_2, where the body S_2 can rotate about its center of inertia C_2, which is fixed in S_1. The system is assumed to be isolated, which means that both the principal vector of the external forces $\mathbf{V}$ and the principal moment of the external forces $\mathbf{m}^\circ$ about an arbitrary point O, are equal zero. When such a system moves in an inertial system of coordinates its principal vector $\mathbf{Q}$ and its principal vector of angular momentum $\mathbf{K}^\circ$ remain invariant.

Let us consider two kinds of motion; in the first one (1+2) the bodies S_1 and S_2 form a single body, in the second one (2) the body S_2 activated by internal forces rotates with respect to S_1. Under the above conditions we have the following theorem: the difference of the kinetic energies

$$T_{(2)} - T_{(1+2)} \geqslant 0 \tag{1}$$

In other words, to maintain the initiated motions we must expend energy, hence the notion that we can transfer the energy from the body S_1 to the body S_2 and utilize it for useful work is faulty (*).

The proof is based on direct calculation of the difference (1) when $\mathbf{Q}$ and $\mathbf{K}^\circ$ remain constant. Let $\mathbf{r} = OC_1$ be the radius vector of the center of inertia C_1 of the body S_1; $\mathbf{v}$ be the velocity vector of C_1; $\boldsymbol{\omega}$ be the angular velocity vector of the body S_1. Then

$$\mathbf{Q}_1 = m_1\mathbf{v}, \qquad \mathbf{K}_1^\circ = \mathbf{r}\times\mathbf{Q}_1 + \Theta^{C_1}\cdot\boldsymbol{\omega}, \qquad 2T_1 = m_1v^2 + \boldsymbol{\omega}\cdot\Theta^{C_1}\cdot\boldsymbol{\omega} \tag{2}$$

where m_1 is the mass, Θ^{C_1} is the inertia tensor of S_1 at the point C_1. Calling $\rho = C_1C_2$ the radius vector of the point C_2 with respect C_1 we have also

$$\mathbf{Q}_2 = m_2(\mathbf{v} + \boldsymbol{\omega}\times\boldsymbol{\rho})$$

$$\mathbf{K}_2^\circ = (\mathbf{r}+\boldsymbol{\rho})\times\mathbf{Q}_2 + \Theta^{C_2}\cdot(\boldsymbol{\omega}+\boldsymbol{\omega} \tag{3}$$

$$2T_2 = m_2|\mathbf{v}+\boldsymbol{\omega}\times\boldsymbol{\rho}|^2 + (\boldsymbol{\omega}+\boldsymbol{\omega}_r)\cdot\Theta^{C_2}\cdot(\boldsymbol{\omega}+\boldsymbol{\omega}_r)$$

where $\boldsymbol{\omega}_r$ is the angular velocity vector of the body S_2 with respect to S_1. Using the relation $\mathbf{Q}_1+\mathbf{Q}_2=\mathbf{Q}$ we can determine $\mathbf{v}$, hence $\mathbf{Q}_1$ and $\mathbf{Q}_2$. After that we find

$$\mathbf{K}^\circ = \mathbf{K}_1^\circ + \mathbf{K}_2^\circ = \mathbf{R}\times\mathbf{Q} + \Theta_1\cdot\boldsymbol{\omega} + \Theta_2\cdot(\boldsymbol{\omega}+\boldsymbol{\omega}_r) \tag{4}$$

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*) This note resulted from the study of a proposed invention of an engine designed to use the energy of Earth's rotation through a certain gyroscopic contraption.

$$2T_{(2)} = 2\,(T_1 + T_2) = \frac{Q^2}{m_1 + m_2} + \omega\cdot\Theta_1\cdot\omega + (\omega + \omega_r)\cdot\Theta_2\cdot(\omega + \omega_r) \tag{5}$$

Here

$$\Theta_1 = \Theta^{c_1} + \frac{m_1 m_2}{m_1 + m_2}(E\rho\cdot\rho - \rho\rho), \qquad \Theta_2 = \Theta^{c_2}, \qquad \mathbf{R} = \mathbf{r} + \frac{m_1 m_2}{m_1 + m_2}\rho \tag{6}$$

(R is the radius vector OC of the center of inertia C of the whole system, E is a unit tensor, $\rho\cdot\rho$ is the scalar product, $\rho\rho$ is the dyadic product).

Denoting the angular velocity vector of the system S_1, S_2 by ω°, when $\omega_r = 0$, we have

$$\mathbf{K}^c = \mathbf{K}^\circ - \mathbf{R}\times\mathbf{Q} = (\Theta_1 + \Theta_2)\cdot\omega^\circ = \Theta_1\cdot\omega + \Theta_2\cdot(\omega + \omega_r) \tag{7}$$

and further

$$2\,(T_{(2)} - T_{(1+2)}) = \omega.\,(\Theta_1 + \Theta_2)\cdot\omega + \omega_r\cdot\Theta_2\cdot\omega_r + 2\omega\cdot\Theta_2\cdot\omega_r - \omega^\circ\cdot(\Theta_1 + \Theta_2)\,\omega^\circ \tag{8}$$

But according to (7) (*)

$$\omega^\circ - \omega = (\Theta_1 + \Theta_2)^{-1}\cdot\Theta_2\cdot\omega_r = \omega_r\cdot\Theta_2\cdot(\Theta_1 + \Theta_2)^{-1} \tag{9}$$

Here $(\Theta_1 + \Theta_2)^{-1}$ is the tensor with the matrix of components, which equals the inverse matrix of components of the tensor $\Theta_1 + \Theta_2$. Using (7) and (8) and performing substitutions in (8) we obtain

$$\begin{aligned} 2\,(T_{(2)} - T_{(1+2)}) &= \omega_r\cdot\Theta_2\cdot\omega_r - (\omega^\circ - \omega)\cdot(\Theta_1 + \Theta_2)\cdot(\omega^\circ - \omega) = \\ &= \omega_r\cdot[\Theta_2 - \Theta_2\cdot(\Theta_1 + \Theta_2)^{-1}\cdot\Theta_2]\cdot\omega_r = \omega_r\cdot\mathbf{Q}\cdot\omega_r \end{aligned} \tag{10}$$

where $\mathbf{Q}$ is the tensor shown in the brackets. In matrix notation we have

$$\mathbf{Q} = \Theta_2 - \Theta_2(\Theta_1 + \Theta_2)^{-1}\Theta_2, \qquad (\Theta_1 + \Theta_2)\,\Theta_2^{-1}\mathbf{Q} = \Theta_1 + \Theta_2 - \Theta_2 = \Theta_1 \tag{11}$$

and further

$$\mathbf{Q} = (\Theta_1^{-1} + \Theta_2^{-1})^{-1}$$

and, since Θ_1, Θ_2 are positive definite matrices, such must be also $\mathbf{Q}$. The theorem is proved. It is valid independently of the size of the bodies, of their angular velocities, and also of the character of the forces of mutual action between the bodies. The simple formula which follows is of interest

$$T_{(2)} - T_{(1+2)} = {}^1/_2\omega_r\cdot(\Theta_1^{-1} + \Theta_2^{-1})^{-1}\cdot\omega_r \tag{12}$$

Translated by T.L.

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*) In matrix notation these equations are written in the form

$$\omega^\circ - \omega = (\Theta_1 + \Theta_2)^{-1}\Theta_2\omega_r, \qquad (\omega^\circ - \omega)' = \omega_r'\Theta_2(\Theta_1 + \Theta_2)^{-1}$$

where prime denotes a transpose of a matrix.